

7\Uh; DH'dfc[bcgh]n]Yfh', '8]b[Yž'X]Y'V]g &\$''a]h'?fmdhc'dUgg]YfYb'k YfXYb

CVk c\``U``Yg`a]h`XYf`; fØbXi b[`XYg`6]hWc]b!BYhnk Yf_g`]a

J\f`&\$\$-`VY[Ubbžk i fXY`X]Y`?fmdhck À\fi b[gVfUbW\Y
]bg[YgUa h'j cf`Yhk U`Y]bYa `J\fnY\bh`[Y[fØbXYh'`K À\fYbX
]b`X]YgYa `NY]hfUi a `g]W\Yf`]W\`bcW\`a Y\f`8]b[Y`dUgg]Yfh
g]bXžX]Y`k]f`\]Yf`Yf_`ÀfYb`_ObbYbž``c\bh`Yg`g]W\`Ui W\ž`g]W\

j cfni ghY``Yb`i bX`XUfØVYf`ni `gdY_i`]YfYbžk Ug`]b`XYb
bÀW\ghYb`nY\b`J\fYb`dUgg]YfYb`_ObbhY"`5i g`X]YgYa ; fi bX
\UVYb`k]f`XYb`Ài »Yfgh`VY`]YVhYb`7\UhVch`ZØf`_Øbgh`]W\Y

-bhY``][Ybn``7\Uh\ DH``bUW\ `gY]bYf`A Y]bi b[`ni `X]YgYa

H\Ya U`VYZfU[h'i bX`Yf``]YZYfhY`Y]b][Y`ØVYfnYi [YbXY
GnYbUf]Yb`XUfØVYfžk Ug`]b`XYb`bÀW\ghYb`nY\b`J\fYb`]b`XYf
?fmdhck Y`h`dUgg]YfYb`_ObbhY"`%"; fO`»YfY`5_nYdhUbn`]a

A U]bghfYUa `8Ug`K]W\h][ghY '/ Ua d\Y``]d/



CVk c\``U``Yg`a]h'XYf`; fØbXi b['XYg`6]hWc]b!BYhnk Yf_g`]a '\$J\f &\$\$- 'VY[Ubbž'k i fXY'X]Y '?fmdhck À\fi b[gVfUbW\Y']bg[YgUa h'j cf Yhk U'Y]bYa '\$J\fnY\bh'[Y[fØbXYh'' K À\fYbX']b'X]YgYa 'NY]hfUi a g]W\Yf`]W\'bcW\'a Y\f'8]b[Y'dUgg]Yfh'g]bXž'X]Y'k]f'\]Yf'Yf_`ÀfYb_\ObbYbž``c\bh'Yg`g]W\'Ui W\ž'g]W\'j cfni ghY``Yb'i bX'XUfØVYf'ni

5i g'X]YgYa ; fi bX'\UVYb'k]f'XYb'Ài » Yfgh'VY`]YVhYb'7\UhVch'ZØf _Øbgh`]W\Y'=bhY``][Ybn' '7\Uh; DH' 'bUW\'gY]bYf'A Y]bi b['ni X]YgYa 'H\Ya U'VYZfU[h'i bX'Yf'`]YZYfhY'Y]b][Y'ØVYfnYi [YbXY GnYbUf]Yb'XUfØVYfž'k Ug']b'XYb'bÀW\ghYb'nY\b'*J\fYb']b'XYf ?fmdhck Y`h'dUgg]YfYb'_ÒbbhY"

%"; fò» YfY 5_nYdhUbn]a A U]bghfYUa

8 UgʻK]WX h][ghYʻni Yfgh. '7 \ Uh; DHʻg_]nn]YfhYʻY]bYʻ [fÒ» YfY
A U]bghfYUa !5_nYdhUbn žʻX]Yʻ]bbYf\U`VʻX]YgYgʻNY]hfU\a Ybg
YfZc`[Ybʻk YfXYžʻXUʻYgʻg]WXʻj cfghY``YžʻXUggʻ6H7ʻi bXʻX]Yʻ5`hWc]bg
Y]bʻk YgYbh`]WXYfʻHY]``XYgʻ[`cVU`Ybʻ:]bUbnbYhnk Yf_gʻk YfXYbʻi bX
ZØfʻHfUbgU_h]cbYbʻj Yfk YbXYhʻk YfXYb''

K]f`\UVYb`]b`XYb``YhnhYb`&J\fYb`g]W\Yf`]W\`9bhk]W_`i b[Yb`Ub X]YgYf`: fcbh`[YgY\Yb"Nk Y]`XYf`;][UbhYb`Ui Z`X]YgYa ; YV]Yh` J]gU`i bX`A UghYfWUfX` `\UVYb`VY]XY`a Y\fYfY`DUfhbYfgW\UZhYb`a]h ?fmdhck À\fi b[gi bhYfbY\a Yb`i bhYfnY]W\bYhž`i a `]\fYb`A]``]cbYb <ÀbX`Yfb`X]Y`5bbU\a Y`j cb`NU\`i b[Yb`ØVYf`X][]hU`Y J Yfa Ò[Ybgk YfhY`ØVYf`]\fY`?fYX]h!`i bX`8YV]h_UfhYb`ni Yfa Ò[`]W\Yb"

 $\label{thm:converse} Hfchn: XYf: FØW_gW. `A[Y:]b: XYf: 6fUbW.Y:]a : `YhnhYb: $J \ f: VY_fAZh][hY XYf: ?fmdhc! 7 \ YZ: j cb: J]gU: _Øfn`]W. `X]Y: dcg]h]j Yb: 5bg]W. hYb: gY]bYg I bhYfbY\a Ybg: ni : X]YgYa : H\Ya Už: XU: g]Y: XUj cb: ØVYfnYi [h: g]bXž XU: g: VYgh]a a hY: X][]hU: Y: 5ggYhg: XU: g: DchYbn]U: `\UVYbž: Y]bY k]W. h][Y: Fc: `Y:]a : NU\`i b[gO_cgmghYa : ni : gd]Y: Yb "$

A UghYfWUfX:\]b[Y[Yb:\Uh:_Øfn`]W\:Y]b:gdYn]Y``Yg:Dfc[fUa a Y]b[YZØ\fhž:XUg:6Ub_Yb:VY]:XYf:9]bZØ\fi b['j cb'?fmdhck À\fi b[g! <UbXY`gd`UhhZcfa Ybii bhYfghØhnYb:gc``"

A UghYfWUfX!; YVÀi XY".7ca di hYfK cf`X

&"'9fk Y]hYfhY'fY[i`Uhcf]gWXY

FU\a YbVYX]b[i b[Yb

8U'X]Y'5_nYdhUbn'ni b]a a hž'dfc[bcgh]n]Yfh'XYf'? #7\UhVchž'XUgg
[`cVU`Y'5i Zg]W\hgVY\OfXYb'X]Y'9bhk]W_`i b['XYf'6fUbW\Y'[YbUi Yf
]a '5i [Y'VY\U`hYb'k c``Yb"'8]Yg'k Uf']b'XYb'`YhnhYb'\J\fYb'Y]b
\Y]» Yg'H\Ya Už']bgVYgcbXYfY'bUW\'XYb'cVYb'Yfk À\bhYb
FØW_gW\`A[Yb']a '`YhnhYb'\J\f' 'U`gc'XYa' Ni gUa a YbVfi W\'j cb
HYffUž': HL 'i bX'XYf'XUfUi ZZc`[YbXYb'5bghYW_i b["

8]Y'JYfY]b][hYb'GhUUhYb'ghUbXYb'Ub'j cfXYfghYf': fcbhžk Ybb'Ui W\b]W\h']b'Y]bYa 'gY\f'dcg]h]j Yb'@]W\h''CVk c\``Xcfh'Y]b][Y'XYf [fÒ»hYb'?fmdhci bhYfbY\a Yb'UbgÀgg]['g]bXž'\UVYb'X]Y'`c_U`Yb FY[i`]Yfi b[gVY\ÒfXYb'Yg'j YfgÀi a hž'Ub[Ya YggYbY'FY[Y`b ZYghni`Y[Yb"

; `Y]WXnY]h][`\Uh'X]Y`GYW f]h]Yg`UbX`9I WXUb[Y`7ca a]gg]cb
a Y\fYfY`:]fa Yb'j YfZc`[hi bX']\bYb'j cf[Yk cfZYbž'b]WXh
fY[]ghf]YfhY`K YfhdUd]YfY`Ubni V]YhYbž'c\bY`XYhU]``]Yfh`Ubni [YVYbž
k Y`WXY`X][]hU`Yb`J Yfa Ò[Ybgk YfhY`hUhgÀWX`]WX`K YfhdUd]YfY`g]bX"
8]Y`FY[i `]Yfi b[gVY\ÒfXY`]gh'Ui WX`]b`FYWXhgghfY]h][_Y]hYb`[Y[Yb
a Y\fYfY`?fmdhc!;][UbhYb'j Yfk]W_Y`hž'XUfi bhYf`F]dd`Y`i bX
; fYmgWU`Y"

GYW f]h]Yg'UbX'9I W\Ub[Y'7ca a]gg]cbž'G97ž'; YVÀi XY']b K Ug\]b[hcb'87"'8]Y'G97'fY[i `]Yfh'5_h]Yb'i bX'5b`Y]\Yb'gck]Y XUa]h'j YfVi bXYbY':]bUbnU_h]j]hÀhYb"

=a ; Y[YbgUhn'XUni ii bhYfbY\a Yb'G]b[Udi fž'<cb[_cb[ii bX'X]Y 9i fcdÀ]gW\Y'I b]cb'GW\f]hhY'ni f'FY[i `]Yfi b['XYf'6fUbW\Y''8]Y'9I\Uh'_Øfn`]W\'X]Y'A]75!; YgYhn[YVi b['j YfUVgW\]YXYhž'X]Y j cfUi gg]W\h`]W\'&\$&('ZØf'GhUV`YWc]bg'i bX'&\$&) 'ZØf'6ÒfgYb'i bX UbXYfY'6fUbW\YbhY]`bY\a Yf']b'?fUZh'hfYhYb'\k]fX"

' "'9f\Ò\hY']bgh]hi h]cbY``Y'=bj Ygh]h]cbYb

K]f`\UVYb`j cf`Y]b][Yb` $\frac{1}{2}$ \fYbž`X]fY_h`bUW\`XYa `5i gVfi W\`j cb 7 C J =8!%- ž`Y]bYb`9]bV`]_`]b`\\\\\\\]bYb\\\GY]hi h]cbYb`Yf\U`hYbž`X]Y`X]YgYb 6YfY]\\\`VYhfUhYb\\'5i Z`XYf`Y]bYb`GY]hY`VfUW\hYb`nU\`fY]\\\Y

dfca]bYbhY'5`h]bj YghcfYb'k]Y'6Yb'A]``Yfž'GhUb'8fi W_Yba]``Yf'i bX DUi ``Hi Xcf' \approx bYg' \Longrightarrow]\fY'I bhYfghØhni b['ZØf'6]hWc]b'i bX'Y]b][Y 5`hWc]bg'ni a '5i gXfi W_ž'k À\fYbX'g]Y'UbXYfY'5b`U[Ycdh]cbYb'i bX X]Y'; Y`Xdc`]h]_'XYf': YX'_f]h]g]YfhYb"

5i Z'XYf'UbXYfYb'GY]hY'\UVYb']bgh]hi h]cbY``Y'5b`U[YjY\]_Y`k]Y X]Y'f]Yg][Y'JYfg]W\Yfi b[g[YgY``gW\UZh'A UggA i hi U`ž'CbY'F]j Yf 5ggYh'A UbU[Yf'i bX'gc[Uf'6`UW_FcW_'fKYf'k Y`hk Y]h'[fÒ» hY JYfa Ò[Ybgj Yfk U`hYfŁ'a]h'VYgh]a a hYb'=bj Ygh]h]cbYb'cXYf X]fY_hYb'6H7!?Ài ZYb']\f'=bhYfYggY'[Yk YW_h"

8]Yg'j YfUb`UgghY'gc[Uf'XYb'Y]bgh'nk Y]ZY`\UZhYb'6Ub_Yb[][UbhYb DA cf[Ub'ni 'XYf'Dfc[bcgYž'XUgg'UbXYfY' + bgh]hi hY']f[YbXk UbbbUW\n]Y\Yb'_ObbhYb"'8]Yg'AbXYfhY'g]W\'YXcW\'YhnhYg'\J\f' 'YJžG]Y'\UVYb'Yg'YffUhYb' 'bUW\'XYa 'Ni gUa a YbVfi W\ž'XYf'a]h'HYffUVY[Ubb'ibX'g]W\'a]h': HL'ZcfhgYhnhY"

»D'A cf[Ub

 $7\ \ DH'[\ `Ui\ Vh'\ YXcW.\ z'XUgg'X]Y'7\ \ UbW'[\ fc\ \ `]ghz'XUgg'g]W.\ `XUg 6`Uhh'k]YXYf'k YbXYhz']bgVYgcbXYfY'k Ybb'X]Y'VY]XYb'j cf\Yf][\ Yb\ J cf\YfgU[\ Yb'k \ U\f'k YfXYb"$

(#) ` '=bhYfU_h]cb'a]h':]bUbnX]Ybgh`Y]ghi b[YbibX'6`cW_WXU]b!9]bZØ\fi b[

K]f'k YfXYb'X]YgY'VY]XYb'Ui Z[fi bX']\fYf'5fh'i bX'XYf'Ÿ\b`]W_Y]h a]h'XYa 'YfghYb'Di b_h'X]YgYf'@]ghY'_ca V]b]YfYb"'7\Uh; DH VY\Ui dhYhYž'XUgg'bYVYb'XYf'Bi hni b['j cb'?fYX]h!'i bX 8YV]h_UfhYb'Ui W\'?fmdhck À\fi b[Yb'gck]Y'X]Y'6`cW_W\U]b! HYW\bc`c[]Y'YfZc`[fY]W\']b'UbXYfY'HY]`Y'XYg':]bUbngmghYa g]bhY[f]Yfh'k YfXYb'_ObbhYb"

8 Uni '[Y\Ofh'X]Y'5Vk]W_`i b['[fc»Yf'HfUbgU_h]cbYb']b'[fO`»YfYa I a ZUb['ØVYf'6`cW_WXU]b!BYhnkYf_Yž'UVYf'Ui WX'X]Y'\deltahY[fUh]cb j cb'JYfg]WXYfi b[gdc`]WYb'i bX'\delta a cV]`]Yb[YgWXÀZhYb'ni gÀhn`]WX ni 'X]YgYf'HYWXbc`c[]Y"

@Ui h'XYa '? #7\UhVch']gh'XUg':]bUbnk YgYb'YXcW\'bi f'Y]bYf'XYf
6YfY]W\Yž']b'XYbYb'6`cW_W\U]b']a a Yf'\Ài Z][Yf'Y]b[YZØ\fh
k YfXYb'_ObbhY"'A Ub'[Y\h'XUj cb'Ui gž'XUgg'X]Y'HYW\bc`c[]Y']b
a Y\fYfYb'UbXYfYb'6YfY]W\Yb'Y]b[YgYhnh'k YfXYb'_ObbhYž'Yhk U
]a '@]YZYf_YhhYba UbU[Ya Ybhž']a '; Ygi bX\Y]hgk YgYb'i bX'gc[Uf
VY]'FY[]Yfi b[gX]YbghYb"

* " B UW \ U `h][_Y]h

#bj Ygh]h]cbYb']b'XYb'6YfY]WXYb'I a k Y`hž'Gcn]U`Yg'i bX; cj YfbUbW'fPG; Ł'k UfYb'j cf'Y]b'dUUf'*J\\fYb']b'U``Yf'A i bXYŽ'U`g Y]b][Y'[`cVU`Y'5[Ybhi fYb'\Ài Z]['VYf]WXhYhYbž'XUgg'6]hWc]b!A]b]b[gWXÀX`]WX'ZØf'X]Y'I a k Y`h'gY]"'G]Y'\UVYb'Yg'[YgWXUZZhž'U`g a Y\fYfY'*bj YghcfYbž'XUfi bhYf'9`cb'A i g_g'HYg`Už'g]WX'j cb'XYb6H7!Dcg]h]cbYb'ni fØW_nc[Yb'i bX'X]Y'A]bYf'i bhYf'8fi W_'gYhnhYbžX]Y'JYfk YbXi b['j cb'?c\`Y'U`g'6fYbbghcZZ'Y]bni ghY``Yb'i bX'g]WXbUWX\U`h][YfYb'i bX'i a k Y`hZfYi bX`]WXYfYb'9bYf[]Yei Y``Ybni ni k YbXYb"

 $8] Yg'] gh'Ui WX'Y] bYf'XYf'; fØbXYž'k Ufi a '9h\YfYi a 'H\Y'A Yf[Y YhnhYg' \(\) \\ f'UV[YgWX \cggYb'\Uh' 'Y] bY'9bhk \] \\ W_i b[\\ z'X] Y'Yg'XYf nk Y] h[fÔ» hYb'6 \cW_WXU] b'Yfa \\ O[\cdot] WXhY\\ z'j ca 'YbYf[]Y] bhYbg] j YfYb \\ DfccZ'cZ'K cf_'ni a 'DfccZ'cZ'GhU_Y' ØVYfni [Y\Yb'' \)$

7\Uh; DH'[Y\h'XUj cb'Ui gž'XUgg'BUW\\U`h][_Y]h']a 'bÀW\ghYb
\$J\fnY\bh'Y]bY'[fc>Y'Fc``Y']b'XYf'6fUbW\Y'gd]Y`Yb'k]fXž'XU'X]Y
I a k Y`hVYXYb_Yb'k Y]hYf'ni bY\a Yb"'8U\Yf'dfc[bcgh]n]YfhY'XYf
7\UhVchž'XUgg'?fmdhck À\fi b[Ybž'X]Y'a Y\f'[fØbY'9bYf[]Y
bi hnYbž'X]Y'VYj cfni [hY'K U\`'ZØf'9G;!\Dj YghcfYb'fi bX'UbXYfYŁ
gY]b'k YfXYb"

+"'8Y:]!9fk Y]hYfi b[

8 UgʻXYnYbhfU`Yʻ:]bUbnk YgYbʻk]fXʻU`gʻ; Y[YbdUfhY]ʻXYg U``[Ya Y]bYbʻ:]bUbngmghYa gʻUb[YgY\Ybži bhYfʻXYa 'X]YʻK Y`h \Yi hYʻZ b_h]cb]Yfhʻi bXʻXUgʻj c``ghÀbX]['nYbhfU`]g]Yfhʻ]gh" '8 Ug VYXYi hYhžʻXUggʻUbʻU```=\ fYbʻHfUbgU_h]cbYbžʻ; YgWkÀZhYbʻi gk " |a a YfʻY]bʻ8f]hhYfʻVYhY]`][hʻ]gh" CVk c\``Yg'8Y:]'gW\cb'gY]h'\J\fYb'[]Vh\z'YfZfYi h'Yg'g]W\']a '\J\f &\$\&\$']a a Yf'[f\overline{O}\rightarrow YfYf'6Y\`]YVh\Y]h"'Dfc Y_hY'Ui g'X]YgYf'B]gW\Y\z'X]Y U\`g' 8Y:]!Gca a Yf 'VYnY]W\bYh'k YfXYb\z'gh]Y[Yb']b'6Yni ['Ui Z 9b[U[Ya Ybh\z'HchU\`JU\`i Y'@cW_'fHJ@\`i bX'DfY]gY'gdfi b[\UZh'Ub"

8]YgY'a Ugg]j Y'i bX'gW\bY``Y'9]bZØ\fi b['\Uh'YXcW\'Ui W\'Y]b][Y DfcV`Ya Y'XYf'6fUbW\Y'ni hU[Y'[YZÒfXYfhž'k]Y'n"'6"'a Ub[Y`bXY G_U`]YfVUf_Y]hž'XUg'J cf\UbXYbgY]b'Y_`UhUbhYf'6YhfØ[YfY]Ybž VYhfØ[Yf]gW\Yf'DfcY_hY'i bX'FUi V_cd]Yb'gck]Y G]W\Yf\Y]hgdfcV`Ya Y"'8i fW\'FUnn]Yb'cXYf'<UW_YfUb[f]ZZY'g]bX']bXYb'`YhnhYb'\J\\fYb'?fmdhck À\fi b[Yb']a 'K Yfh'j cb'a Y\fYfYb A]``]UfXYb'j Yf`cfYb'[Y[Ub[Yb"

B]WkhgXYghchfchn'[`UiVh'7\Uh; DHžXUgg'8Y:]'gY]bY'DfcV`YaY ØVYfghY\YbžXUg'6YbihnYfYf`YVb]g'jYfVYggYfb'ibX'ni'Y]bYak]Wkh][Yb'6YghUbXhY]`'XYf':]bUbnVfUbWkY'kYfXYb'k]fX"

, "'BYi Y'hYWXbc`c[]gWXY'=bbcjUh]cbYb

Ni `[i hYf'@Yhnh`Yfk À\bhY`XYf`? \pm 7\UhVch` \pm bcj Uh]cbYb`]a `HYW\! 6YfY]W\"`GW\`]Y» `]W\']gh`XUg`6]hWc]b!BYhnk Yf_ž'k]Y`Ui W\'j]Y`Y UbXYfY`Dfc Υ _hYž'X]Y`]b`XYb`Zc`[YbXYb`%) ` \pm U\fYb`YbhghUbXYb`g]bXž Y]bY` \pm bcj Uh]cb`ZØf`g]W\"

Hfchn'gY]bYf'DfcV`Ya Y'ZØ\fhY'XYf'DfccZ'cZ'K cf_'ni f'GW\UZZi b[XYg'DfccZ'cZ'GhU_Y'i bX'gY]bYf'nU\`fY]W\Yb'J Uf]Uh]cbYb"'6]g\Yf\UVYb'k]f'[YgY\Ybž'XUgg'6`cW_W\U]b'X]Y'9bhk]W_`i b['nU\`fY]W\Yfi bX'i bhYfgW\]YX`]W\Yf'Dfchc_c``Y'a]hi bhYfgW\]YX`]W\Yb'Nk YW_YbYfa Ò[`]W\hž'k]Y'ni a '6Y]gd]Y`'8Y:]ž'B: Hgž'CfU_Y`'i gk''

8]Y'bÀW\ghYb'nY\b'\\J\fY'_ObbhYb'j]Y``a Y\f'Bi hni b['ZØf'\\XYYbVf]b[Ybž'Ub'X]Y'k]f'bcW\'b]W\h'Y]ba U``[YXUW\h'\UVYb"

9g'_ÒbbhYb'bYi Y'?fmdhck À\fi b[Yb'a]h: i b_h]cbYb'i bX

J Yfk YbXi b[ga Ò[`]W_Y]hYb'YbhghY\Ybž'X]Y'k]f'XYfnY]h'b]W\h
j cf\YfgY\Yb'_ÒbbYb"'8]YgY'_ÒbbhYb'Xi fW\'hYW\bc`c[]gW\Y
: cfhgW\f]hhY'cXYf'j YfÀbXYfhY'6YXØfZb]ggY'i bX'6YX]b[i b[Yb']b

XYf'; YgY``gW\UZh'VYX]b[h'gY]b"

. . 11

8 YhUl`d

6Ygi WXYb G]Y i bg Ui Z _fmdhc!bYk g"Uh